International Journal of Theoretical Physics, Vol. 36, No. 8, 1997

A Transformation Method of Generating Exact Analytic Solutions of the Schrödinger Equation

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Received January 27, 1997

A transformation method is presented which consists of a coordinate transformation and a functional transformation that allow generation of normalized exact analytic bound-state solutions of the Schrödinger equation, starting from an analytically solved quantum problem. The coordinate transformation is the basic transformation, which is supplemented by the functional transformation so that one can choose the dimension of the space of the transformed system. By repeated application of the method, it is possible to generate a number of solved quantum problems in the case that the original quantum system has a multiterm potential. It is shown that the eigenfunction of the transformed system can be easily normalized in most cases.

1. INTRODUCTION

Besides the exact analytic solutions (EAS) of the Schrödinger equation, which are few and far apart, it is the approximate perturbation schemes and variational procedures which give quantum mechanics its enormous success as a physical theory. For implementing approximate schemes economically and profitably while dealing with practical quantum mechanical problems, EAS of idealized quantum systems (QS) are desirable. Nonperturbative solutions of different potentials may lead to new physical ideas and/or calculational techniques in quantum physics. In fact there has been a sustained effort in this direction (Manning, 1935; Biswas *et al.*, 1971; Khare, 1981; Roy and Roychoudhury, 1987; Chhajlany and Malnev, 1990; Bose, 1994), with a recent increase in activity mainly inspired by supersymmetric quantum mechanics (Cooper *et al.*, 1995). This provides the raison d'être for the search for other solvable quantum systems. The intention of this paper is to elucidate a simple mapping procedure for generating normalized bound-state EAS of the

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0020-7748/97/0800-1893$12.50/0

Schrödinger equation for new potentials, albeit generated from an already known normalized EAS.

Johnson (1980) and Dutra (1988, 1993) gave some mapping procedures, but the aim and content of their procedures are different from what is presented in this paper. In our scheme we start from a known analytically solved QS and transform it to generate a new solved QS. When the original QS has a multiterm potential, then in general it is possible to generate a number of different solved QS, depending on the number of ways of grouping the various terms of the potential. Besides the normal exactly solvable QS, other analytically solvable QS include the quasi-exactly solvable QS (Flessas, 1979; Flessas and Das, 1980; Dutra, 1988; Shiffman, 1989; Dutra and Filho, 1991), conditionally exactly solvable QS (Dutra 1993; Dutta *et al.*, 1995a), and conditionally quasi-exactly solvable QS (Dutta *et al.*, 1995b). In our procedure the new QS preserves the nature of the original QS from which it is generated.

The transformation procedure is based on a coordinate transformation (CT), but is found to be inadequate, as it leads to problems regarding the dimensionality of the (Euclidean) space into which the transformed system gets transported. This is illustrated by applying the CT to the often discussed mapping of a Coulomb to a harmonic oscillator system. This problem can be overcome by supplementing the CT by a functional transformation (FT) that allows a consistent way to choose the dimension of the transformed system. The generated bound-state QS are deemed solved only when the energy eigenstates are normalizable. In this transformation method the normalizibility of the eigenfunctions of the generated QS can easily be verified in most cases.

The organization of the paper is as follows. In Section 2, we give the transformation procedure to generate the EAS for new potentials, considering the coordinate transformation. Section 3 applies the method to the Coulomb and harmonic oscillator problems, including the normalizibility of the generated EAS. An extended transformation which combines the coordinate transformation and functional transformation is presented in Section 4 along with the reasons for its necessity. The important question of the normalizibility of the generated EAS is discussed in Section 5, and conclusions are given in Section 6.

2. COORDINATE TRANSFORMATION

Consider a solved quantum bound-state problem in the nonrelativistic regime, henceforth called the A system, in D_A-dimensional Euclidean space.

The radial Schrödinger equation (Louck, 1960) is ($\hbar = 1 = 2m$)

$$\psi_A''(r) + \frac{D_A - 1}{r}\psi_A'(r) + \left(E_n^A - V_A(r) - \frac{l_A(l_A + D_A - 2)}{r^2}\right)\psi_A(r) = 0 \tag{2.1}$$

where the normalized eigenfunction $\psi_A(r)$ and the energy eigenvalues E_n^A are presumed known for the given central potential $V_A(r)$. The prime denotes differentiation with respect to the argument.

Consider the coordinate transformation (CT)

$$r \to g_B(r)$$

with the transformation function $g_B(r)$ a differentiable function of at least class C^2.

To implement the program it is necessary to make the following ansätze, which constitute an integral part of the transformation method:

$$g_B'^2 V_A(g_B) = -E_N^B \tag{2.2}$$

$$g_B'^2 E_n^A = -V_B(r) \tag{2.3}$$

and

$$\frac{g_B'^2(l_A + D_A/2 - 1)^2}{g_B^2} = \frac{(l_B + D_B/2 - 1)^2}{r^2} \tag{2.4}$$

The generalization of the Langer form (Langer, 1937) is taken for the 'centrifugal barrier' term. Then equation (2.1) takes the form

$$\chi_B''(r) + \chi_B'(r)\left(\frac{d}{dr}\ln\frac{g_B^{D_A-1}}{g_B'}\right) + g_B'^2\left(E_n^A - V_A(g_B) - \frac{(l_A + D_A/2 - 1)^2}{g_B^2} + \frac{(D_A - 2)^2}{4g_B^2}\right)\chi_B(r) = 0 \tag{2.5}$$

with $\chi_B(r) = \psi_A(g_B(r))$. With the ansätze (2.2)–(2.4), equation (2.5) takes the form

$$\chi_B''(r) + \chi_B'(r)\left(\frac{d}{dr}\ln\frac{g_B^{D_A-1}}{g_B'}\right) + \left(E_N^B - V_B(r) - \frac{(l_B + D_B/2 - 1)^2}{r^2} + \frac{g_B'^2}{g_B^2}\frac{(D_A - 2)^2}{4}\right)\chi_B(r) = 0 \tag{2.6}$$

which we call the transformed system (B system). The B system would in general be in a Euclidean space of dimension[2] η_B other than D_A. For a power-law potential the specific value of the dimension D_B is dependent on the functional form of $g_B(r)$, which is essentially dependent on the form of $V_A(r)$ as indicated by the first ansatz (2.2). The second ansatz (2.3) specifies the new potential $V_B(r)$ and at the same time yields the energy eigenvalues E_N^B, as shown below. The last ansatz (2.4) gives the required relationship between the 'angular momentum' quantum numbers l_A and l_B which allows us to replace all l_A-dependent quantities of the A system with the corresponding l_B-dependent quantities of the B system. In this method there is no room for any arbitrary choice for the quantities g_B, l_B, D_B, $V_B(r)$, and E_N^B while changing over from the A to the B system.

For a power-law potential $V_A(r) = a_A r^{b_A}$, $g_B(r) = \text{const} \cdot r^{2/(b_A+2)}$, as given by equation (2.2). This gives the coefficient of $\chi_B'(r)$ in equation (2.6) to be

$$\left(\frac{2(D_A + b_A)}{b_A + 2} - 1\right)\frac{1}{r}$$

It is therefore natural to identify $2(D_A + b_A)/(b_A + 2) = \eta_B$ as the dimension of the space in which this B-system Schrödinger equation is established. Further note that this identification of D_B allows the quantity

$$\frac{(l_B + D_B/2 - 1)^2}{r^2} - \frac{g_B'^2}{g_B^2}\frac{(D_A - 2)^2}{4}$$

to be reduced to $l_B(l_B + D_B - 2)/r^2$, as consistency requires. As a result, equation (2.6) becomes

$$\chi_B''(r) + \frac{\eta_B - 1}{r}\chi_B'(r) + \left(E_N^B - V_B(r) - \frac{l_B(l_B + \eta_B - 2)}{r^2}\right)\chi_B(r) = 0 \tag{2.7}$$

which is the B system Schrödinger equation in η_B-dimensional space with the new potential $V_B(r)$ [given by ansatz (2.3)] and whose analytic solution is

$$\chi_B(r) = \psi_A(g_B(r)) \tag{2.8}$$

It should be noted that the dimension of the Euclidean space η_B in which the Schrödinger equation (2.7) is established by a coordinate transformation

[2] The dimension of the Euclidean space in which the Schrödinger equation is established is denoted by D, except when it is a coordinate-transformed Schrödinger equation. The dimension of the coordinate-transformed Schrödinger equation is denoted by η for reasons that will be apparent in the text.

is not arbitrary. η_B is the coefficient of $\chi'_B(r)/r$ in equation (2.6) increased by unity, i.e.,

$$\eta_B = 1 + r\frac{d}{dr}\ln\left(\frac{g_B^{D_A-1}(r)}{g'_B(r)}\right) \tag{2.9}$$

$\eta_B = \eta_B(D_A, b_A)$ for a power-law potential with b_A as the exponent of r. The above expression shows that, given two of the characteristic quantities of the A system Schrödinger equation D_A and b_A, the dimension η_B of the transformed B system is uniquely specified and there is no way it can be chosen *a priori* in a consistent manner.

For $g_B(r)$ a power law, $g'_B(r)$ is also a power-law function of r; hence, by equation (2.3) we get a power law $V_B(r)$. The energy eigenvalues E_N^B of the B system are simply obtained by putting the coefficient of the r-independent part of $V_B(r)$, which would be a function $F(E_N^B)$, equal to the characteristic constant C_B^2 of the B system (Ahmed, 1996), i.e.,

$$V_B(r) = F(E_N^B)E_n^A r^\rho = C_\beta^2 r^\beta \tag{2.10}$$

This yields

$$E_N^B = F^{-1}(C_B^2/E_n^A) \tag{2.11}$$

E_N^B is thus specified in terms of the known E_n^A of the A system. However, the quantum numbers N and n are different, as l_A and l_B are in general different. For the potential $V_A(r) = a_A r^{b_A}$, $2l_A = (b_A + 2)l_B$. The form of $g_B(r)$ decides the form of E_N^B in terms of E_n^A, besides deciding the dimension, which we call the natural dimension η_B of the Euclidean space, where the transformed system (B system) would be found when the mapping of the A to the B system involves only the coordinate transformation.

3. COORDINATE TRANSFORMATION OF COULOMB AND HARMONIC OSCILLATOR SYSTEMS

That a Coulomb system can be mapped into a harmonic oscillator system is not new. However, to our knowledge this has not been completely analyzed. In this section we apply the above procedure to a 3-dimensional Coulomb system and transform it to a 4-dimensional harmonic oscillator (HO) system and also indicate why a 3-dimensional HO cannot be similarly transformed into a Coulomb system by a coordinate transformation.

The 3-dimensional Coulomb system is characterized by (in atomic units: $\hbar = 1, e^2 = 2, m = 1/2$) the potential $V_A(r) = V_{coul} = -2/r$, energy eigenvalue $E_n^A = -1/n_A^2$, $n_A = n_r + l_A + 1$, and radial eigenstate

$$\psi_A(r) = r^{l_A}e^{-r/n_A}L_{n_A-l_A-1}^{2l_A+1}\left(2\frac{r}{n_A}\right) \tag{3.1}$$

A simple integration of equation (2.3) with $V_A(g) = -2/g$ yields

$$\sqrt{g(r)} = C \pm 0.5 \left(\frac{E_N^B}{2}\right)^{1/2} r \tag{3.2}$$

Taking the integration constant $C = 0$, which is equivalent to $g(0) = 0$, and the positive sign in equation (3.2) yields

$$g(r) = (E_N^B/8)r^2 \tag{3.3}$$

leading by equation (2.4), to

$$V_B(r) = -(E_N^{B^2}/16)E_n^A r^2 = (E_N^{B^2}/16n_A^2)r^2 \tag{3.4}$$

$V_B(r)$ is identified as the HO potential. Equating the coefficient of r^2 to the characteristic constant $\frac{1}{4}w^2$ of the HO yields

$$V_B(r) = \frac{\omega^2}{4} r^2 \tag{3.5}$$

and

$$E_N^B = \omega(2n_A) \tag{3.6}$$

Expressions (2.9) and (3.3) and the fact that the original Coulomb system is in 3-dimensional space (i.e., $D_A = 3$) show that $\eta_B = 4$ for the HO system. Equation (2.4) then predicts the relation $2l_A = l_B$ and hence $2n_A = 2n_r + l_B + 2 = N + 2$. Therefore the energy eigenvalue $E_N^B = \omega(N + 2)$, as expected for the 4-dimensional HO. The radial eigenfunction $\chi_B(r)$ given by equation (2.3) becomes

$$\chi_B(r) = r^{l_B} e^{-(\omega/4)r^2} L_{1/2(N-l_B)}^{l_B+1}\left(\frac{1}{2}\omega r^2\right) \tag{3.7}$$

which satisfies the 4-dimensional radial Schrödinger equation

$$\chi_B''(r) + \frac{3}{r}\chi_B'(r) + \left[\omega(N + 2) - \frac{\omega}{4} r^2 - \frac{l_B(l_B + 2)}{r^2}\right]\chi_B(r) = 0 \tag{3.8}$$

A further coordinate transformation on this 4-dimensional HO would be an inverse transform and it would revert back to the 3-dimensional Coulomb case. It is interesting to note, however, that when we start from a 3-dimensional HO system, then the above coordinate transformation procedure yields $\eta_B = 3/2$, showing that the transformed Coulomb system would not be in any integral-dimensional space and hence could not be an ordinary physical Coulomb system. This kind of dimensional mismatch is a deficiency of the coordinate transformation method.

Normalizability of the wave function is an important aspect of a solved bound-state quantum problem. In the present case the B system wave function is normalizable if the integral

$$\int_0^\infty \psi_A^*(g)\psi_A(g)r^3\,dr$$

is finite and positive. Since $g_B(r) = \alpha r^2$, we can write $r^3\,dr$ as $(2/\alpha)$ $\{[-V_A(g)]/E_N^B\}\,g^2dg$, so that the above integral becomes

$$\int_{g(0)}^{g(\infty)} \psi_A^*(r)\,\frac{2}{\alpha}\,\frac{[-V_A(r)]}{E_N^B}\,\psi_A(r)r^2\,dr \to \frac{2}{\alpha E_N^B}\,\langle -V_A(r)\rangle \tag{3.9}$$

which is finite and positive, as E_N^B is never zero, implying normalizibility of the 4-dimensional harmonic oscillator wave function $X_B(r) = \psi_A(\alpha r^2)$ obtained from the 3-dimensional Coulomb system. A general discussion on the normalizibility of the wave function of the transformed system is given in Section 5.

4. EXTENDED TRANSFORMATION

The above simple examples amply demonstrate that it is quite easy to map one type of quantum system to another following the above coordinate transformation method. However, in two important aspects it is deficient. (i) One cannot convert a given quantum system into a desired quantum system, (ii) it is not possible, for a power-law potential, to control the dimension of the space in which the transformed quantum system will be found, as the dimension η_B is uniquely specified. Moreover, the transformed systems cannot be cast in the standard Schrödinger equation form when the potential is non-power law.

The second deficiency can be overcome by performing an extended transformation (ET) instead of the simple coordinate transformation on the A system that we considered above. The extended transformation also solves the problem encountered in applying the coordinate transformation to the Schrödinger equation with a non-power-law potential. The extended transformation consists of a coordinate transformation followed by a functional transformation, as follows:

$$r \to g_B(r) \tag{4.1}$$

and

$$\chi_B(r) = f^{-1}(r)\psi_A(g_B(r)) \tag{4.2}$$

For the extended transformation the transformation function $g_B(r)$ must be a differentiable function of at least class C^3.

Application of the extended transformation to equation (2.1) of the A system gives

$$\chi_B''(r) + \chi_B'(r)\left(\frac{d}{dr}\ln\frac{f^2 g_B^{D_A-1}}{g_B'}\right) + \left[\left(\frac{d}{dr}\ln f\right)\left(\frac{d}{dr}\ln\frac{f' g_B^{D_A-1}}{g_B'}\right)\right.$$
$$\left. + g_B'^2\left(E_n^A - V_A(g_B) - \frac{l_A(l_A + D_A - 2)}{g_B^2}\right)\right]\chi_B(r) = 0 \qquad (4.3)$$

To have the transformed system in a chosen D_B-dimensional Euclidean space, we set

$$\frac{d}{dr}\ln\frac{f^2 g_B^{D_A-1}}{g_B'} = \frac{D_B - 1}{r}$$

which fixes $f(r)$ as

$$f(r) = g_B'^{1/2} g_B^{-(D_A-1)/2} r^{(D_A-1)/2}$$

and changes equation (4.3) to

$$\chi_B''(r) + \frac{D_B - 1}{r}\chi_B'(r) + \left[\frac{1}{2}\{g_B, r\}\right.$$
$$- \frac{D_A - 1}{2}\frac{D_A - 3}{2}\left(\frac{g_B'}{g_B}\right)^2 + \frac{D_B - 1}{2}\frac{D_B - 3}{2}\left(\frac{1}{r^2}\right)$$
$$\left. + g_B'^2\left(E_n^A - V_A(g_B) - \frac{(l_A + D_A/2 - 1)^2}{g_B^2} + \frac{(D_A - 1)^2}{4g_B^2}\right)\right]\chi_B(r) = 0 \qquad (4.4)$$

where

$$\{g_B, r\} = \frac{g_B'''(r)}{g_B'(r)} - \frac{3}{2}\left(\frac{g_B''(r)}{g_B'(r)}\right)^2$$

and is the Schwartzian derivative symbol. Invoking the ansatze (2.2)–(2.4) of Section 2 reduces equation (4.4) to the standard Schrödinger equation form

$$\chi_B''(r) + \frac{D_B - 1}{r}\chi_B'(r) + \left\{E_N^B - V_B(r) - \left[-\frac{1}{2}\{g_B, r\} + \frac{D_A - 1}{2}\frac{D_A - 3}{2}\left(\frac{g_B'}{g_B}\right)^2 - \frac{D_B - 1}{2}\frac{D_B - 3}{2}\left(\frac{1}{r^2}\right) + \frac{(l_B + D_B/2 - 1)^2}{r^2} - \left(\frac{g_B'}{g_B}\right)^2\frac{(D_A - 2)^2}{4}\right]\right\}\chi_B(r) = 0 \tag{4.5}$$

It is interesting to note that the quantities inside the square brackets give the correct form of the 'centrifugal barrier' term in a D_B-dimensional space $l_B(l_B + D_B - 2)/r^2$ (Louck, 1960) whenever $V_A(r)$ is of power-law type. Thus the transformed B system Schrödinger equation is established in a Euclidean space of the chosen dimension D_B, and is given by

$$\chi_B''(r) + \frac{D_B - 1}{r}\chi_B'(r) + \left[E_N^B - V_B(r) - \frac{l_B(l_B + D_B - 2)}{r^2}\right]\chi_B(r) = 0 \tag{4.6}$$

whose eigenfunction is given by equation (4.2):

$$\chi_B(r) = g_B^{l-1/2} g_B^{(D_A-1)/2} r^{-(D_B-1)/2}\psi_A(g_B(r)) \tag{4.7}$$

and is known as $\psi_A(r)$ and $g_B(r)$ are known.

The energy eigenvalues E_N^B are given by equation (2.9). The relation between the 'angular momentum' quantum numbers l_A and l_B obtained from ansatz (2.4) is

$$4l_A = (b_A + 2)(2l_B + D_B - 2) - 2(D_A - 2) \tag{4.8}$$

When we apply the extended transformation to transform a 3-dimensional Coulomb to a HO system, then, unlike the coordinate transformation, we can now choose the dimension of the HO system. Let the HO system be required in 3-dimensional space also. Expressions (3.3), (3.5), and (3.6) for $g_B(r)$, $V_B(r)$, and E_N^B, respectively, hold here, too, except that the relation between the 'angular momentum' quantum numbers l_A and l_B is given by equation (4.8), leading to $l_A = l_B/2 - 1/4$ and so $2n_A = 2n_r + (l_B - 1/2)$

$+ 2 = N + 3/2$. The energy eigenvalue is now $E_N^B = \omega(N + 3/2)$. In addition, instead of $\chi_B(r) = \psi_A(g_B(r))$, we now have [equation (4.7)]

$$\chi_B(r) = r^{-1} g_B(r)'^{-1/2} g_B(r) \psi_A(g_B(r))$$
$$= r^{l_B} e^{-(\omega/4)r^2} L^{l_B+1/2}_{1/2(N-l_B)}\left(\frac{1}{2}\omega r^2\right) \tag{4.9}$$

which is the radial eigenfunction of the 3-dimensional HO. In an exactly similar fashion we can consistently map a 3-dimensional HO system to a Coulomb system of any desired dimension.

5. NORMALIZABILITY OF THE TRANSFORMED WAVE FUNCTION

The normalizability of the B system wave function obtained by the extended transformation can be proved under fairly general conditions, as it seems to preserve the normalizability property to quite a good extent.

Expressions (4.7) and (2.2) give the normalization integral for $\chi_B(r)$ as

$$|N_B|^2 \int_{g_B(0)}^{g_B(\infty)} \psi_A^*(r)\left[-\frac{V_A(r)}{E_N^B}\right]\psi_A(r) r^{D_A-1}\, dr = 1 \tag{5.1}$$

Hence all those $\chi_B(r)$ are normalizable for which (i) $E_N^B \neq 0$ and (ii) the integral

$$I(g_B(\infty), g_B(0)) = \int_{g_B(0)}^{g_B(\infty)} \psi_A^*(r)\left[-\frac{V_A(r)}{E_N^B}\right]\psi_A(r) r^{D_A-1}\, dr \tag{5.2}$$

is positive definite.

It is pertinent to note at this stage that, since $\psi_A(r)$ is the normalized wave function of a genuine quantum mechanical system, the quantity $(-E_N^B)I(\infty, 0) = \langle V_A(r)\rangle$ necessarily exists. Its existence also implies that $(-E_N^B)I(b, a)$ also exists whenever $a > 0$ and $b > a$. The generated eigenfunctions $\chi_B(r)$ are normalizable, therefore, whenever the asymptotic and the local behavior of $g_B(r)$ ensure that $g_B(\infty) > 0$, $g_B(r) \geq 0$, and $g_B(\infty) > g_B(r)$.

6. DISCUSSION AND CONCLUSION

This paper has dealt with the generation of exact analytic bound-state solution of the Schrödinger equation, talking as 'seed' an already analytically solved quantum problem. The method rests on a coordinate transformation followed by a functional transformation. The former is the basic transforma-

tion, and has been used in earlier treatments mapping one system into another. Earlier studies of mapping by various authors dealt with a 1-dimensional system and/or completely sidetracked the important question of the dimensionality of the space of the transformed system. Within the context of the coordinate transformation and power-law potential we have studied the dimensionality aspect of the transformed quantum system and found that except when the A system is taken in 2-dimensional space, there is always a change in the dimensionality of the B system. This uniquely determined number η_B, representing the dimensionality of the B system, is given by equation (2.9), which for the specific form $V_A(r) = a_A r^{b_A}$ becomes

$$\eta_B - 2 = \frac{2(\eta_A - 2)}{b_A + 2} \tag{6.1}$$

The dimensionality η_B (D_A, b_A) is a characteristic quantity of the B system. The uniqueness of η_B causes trouble, as, depending on η_A and b_A, it may even become fractional. For example, if the A system is a one-dimensional HO system ($\eta_A = 1$, $b_A = 2$) or a three-dimensional HO system ($\eta_A = 3$, $b_A = 2$), the B system obtained by coordinate transformation would be a Coulomb system in 3/2 or 5/2 dimensional Euclidean space, respectively. We have to start with a 4-dimensional HO as the A system to get a 3-dimensional Coulomb system by coordinate transformation.

When the exponent of the power-law potentials of the A system and its coordinate transformed B system are b_A and b_B, respectively, then corresponding to relation (6.1) we also have $\eta_A - 2 = 2(\eta_B - 2)/(b_B + 2)$ obtained by interchanging the roles of the A and B systems. These two relations at once give the duality relation

$$(b_A + 2)(b_B + 2) = 4 \tag{6.2}$$

which can also be derived from the ansatz (2.4). It is worth noting that the above relation for $(\eta_B - 2)$, which is valid under the coordinate transformation, reduces the relation (4.8) to

$$l_A(b_A + 2)^{-1/2} = l_B(b_B + 2)^{-1/2} \tag{6.3}$$

As mentioned earlier, the difficulty of not being able to choose the dimension of the transformed B system *a priori* led us to supplement the coordinate transformation by a functional transformation and consider the composite transformation—the extended transformation. The F-transformation component of the E-transformation may be considered as a device for dimensional reduction or dimensional extension of the transformed B system with reference to the dimension η_B.

Let α, which we can choose, be the deviation of the dimensionality of the B-system under the E-transformation compared to η_B, i.e., $D_B = \eta_B + \alpha$. Then relation (6.3) becomes

$$2l_A(b_A + 2)^{-1/2} = (2l_B + \alpha)(b_B + 2)^{-1/2} \tag{6.4}$$

The application of this transformation method to generate different completely solved quantum mechanical problems starting from an analytically solved problem with central power-law potential is fairly straightforward. It is not so when the original quantum system is governed by a potential other than a central power law. In such situations, the n-dependent part and the r-dependent part of the transformation function $g_B(r)$ get intertwined without yielding to factorization. These complexities have been addressed by applying the method to the Hulthen problem, which has a typical non-power-law potential, and will be reported elsewhere.

It may be mentioned that the transformation procedure may be applied repeatedly in cases involving a central multiterm power-law potential by selecting working potentials differently to generate a variety of solved quantum problems, the number, in principle, being equal to the number of ways $(2^n - 1)$ in which the working potential may be chosen when the original quantum system is governed by an n-term potential.

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